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LETTER TO THE EDITOR

The Heisenberg antiferromagnetic chain†

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Abstract. Simple renormalisation group analysis in conjunction with the group-theoretic method is used to prove that the Heisenberg antiferromagnetic chain admits of no phase transition.

The advent of lattice field theory has not only provided a new approach to the study of field theory, but also introduced some novel and effective methods that are applicable to a number of important problems in condensed matter physics. Critical phenomena, spin glass and localisation are some of the notable examples.

Among the new methods invented in the course of studying lattice field theory was the truncation renormalisation group proposed by Drell *et al* (1977). This method represents one of the simplest realisations of the ideas of the renormalisation group block spin transformation. First the spins are grouped into blocks. The Hamiltonian is then rewritten as a sum of intra-block interactions and inter-block interactions. A single intra-block Hamiltonian is solved exactly. The lowest-lying eigenstates are retained to construct a renormalised Hamiltonian that is isomorphic to the original bare Hamiltonian. Recursion relations are thus established between the renormalised and the bare coupling constants. Whether a phase transition exists is determined by the presence or absence of non-trivial fixed points. If non-trivial fixed points do exist, critical eigenvalues, and consequently critical exponents, can be computed by linearisation in the neighbourhood of the fixed points. This truncation renormalisation group has been applied successfully to many interesting models (Jullien *et al* 1978, Hu 1979, 1980a,b,c). It is also simple enough to deal with quite complicated systems in which more elaborate methods are too involved to be manageable. Moreover, it respects local, or gauge, symmetry, which is seminal to the study of field theory, spin glass and other amorphous systems. However, although the truncation renormalisation group is quantum mechanical in nature, application of this method has been mainly concerned with *classical* models reformulated as quantum mechanical models via the transfer matrix. In this note we will apply this renormalisation group to the study of the quantum mechanical Heisenberg antiferromagnetic chain. As we will show, this renormalisation group method, in conjunction with group theory, provides a simple way to prove the well known fact that no phase transition exists for this model. This approach was first employed by Svetitsky (1980) in the study of dynamical symmetry breaking in lattice gauge theories.

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The reduced Hamiltonian of the quantum mechanical Heisenberg antiferromagnetic chain is given as follows:

$$H = K \sum_i \boldsymbol{S}(i)\boldsymbol{S}(i+1), \tag{1}$$

when $K = J/kT$, and $\boldsymbol{S}$'s are the spin-$\frac{1}{2}$ angular momentum operators.

To effect the renormalisation group block spin transformation we group three spins into a block. The Hamiltonian can then be rewritten as

$$H = H_0 + H_1,$$

where

$$H_0 = K \sum_I [\boldsymbol{S}_1(I) \cdot \boldsymbol{S}_2(I) + \boldsymbol{S}_2(I) \cdot \boldsymbol{S}_3(I)]. \tag{2}$$

is the intra-block Hamiltonian, and

$$H_1 = K \sum_I \boldsymbol{S}_3(I) \cdot \boldsymbol{S}_1(I+1)$$

is the inter-block Hamiltonian. 'I' labels the blocks whereas the subscripts label the spins within a block.

We first try to solve H_0 exactly. This can be done easily by solving a single block Hamiltonian h_0 since all intra-block Hamiltonians commute. Only the lowest lying eigenstates are then retained to construct a truncated Hilbert space. Finally the inter-block Hamiltonian is represented in this truncated Hilbert space, giving rise to a truncated renormalised Hamiltonian. Recursion relations are thus established between the original and the renormalised coupling constants.

To solve the single block Hamiltonian exactly, we rewrite it as follows:

$$\begin{aligned} h_0 &= K(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_2 \cdot \boldsymbol{S}_3), \\ &= \tfrac{1}{2}K(\mathbf{S}^2 - \mathbf{S}_{13}^2 - \boldsymbol{S}_2^2), \end{aligned} \tag{3}$$

where

$$\boldsymbol{S}_{13} = \boldsymbol{S}_1 + \boldsymbol{S}_3$$

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3.$$

Since $\boldsymbol{S}^2$ and $\boldsymbol{S}_{13}^2$ commute, they can be diagonalised simultaneously. Diagonalisation of h_0 is then simply a problem of addition of angular momenta. We first add $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ to form states of definite $\boldsymbol{S}_{13}^2$:

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}. \tag{4}$$

We then add $\boldsymbol{S}_{13}$ and $\boldsymbol{S}_2$ to form states of definite $\boldsymbol{S}^2$:

$$\begin{aligned} &\mathbf{1} \times \mathbf{2} = \mathbf{2} \\ &\mathbf{3} \times \mathbf{2} = \mathbf{2} + \mathbf{4}. \end{aligned} \tag{5}$$

To find the representation that contains the lowest-lying eigenstates we have to minimise h_0 since it is an antiferromagnet. Therefore we have to minimise $\boldsymbol{S}^2$ and maximise $\boldsymbol{S}_{13}^2$. Hence the lowest-lying eigenstates we retain are in the doublet representation **2**—the same as the original site-spin, and the symmetry of the antiferromagnet is thus preserved.

The evaluation of the matrix representation of the inter-block Hamiltonian in this truncated basis follows the same procedure as that of the derivation of the Landé g-factor based on the Wigner–Eckart theorem:

$$\langle \mathbf{2}m|\boldsymbol{S}_\alpha|\mathbf{2}m'\rangle = g_\alpha \langle \mathbf{2}m|\boldsymbol{S}|\mathbf{2}m'\rangle,$$

where

$$\begin{aligned} g_1 &= g_3 = \tfrac{2}{3} \\ g_2 &= -\tfrac{1}{3}. \end{aligned} \tag{6}$$

Therefore the truncated Hamiltonian (modulo a constant term) reads:

$$H' = K' \sum_I \boldsymbol{S}(I) \cdot \boldsymbol{S}(I+1) \tag{7}$$

It is completely isomorphic to the original Hamiltonian except for the renormalisation of the coupling constant:

$$K' = \tfrac{4}{9}K. \tag{8}$$

It is now trivial to see that the Heisenberg antiferromagnetic chain admits of no phase transition since there exists no non-trivial fixed point in the linear recursion relation (8).

Application of this method to other interesting quantum mechanical problems is in progress.

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References

Drell S D, Weinstein M and Yankielowicz S 1977 *Phys. Rev.* D **16** 1769–81
Hu B 1979 *Phys. Lett.* **71**A 83–6
—— 1980a *Phys. Lett.* **75**A 372–4
—— 1980b *Phys. Lett.* **79**A 6–8
—— 1980c *J. Phys. A: Math. Gen.* **13** L321–3
Jullien R, Pfeuty P, Fields J N and Doniach S 1978 *Phys. Rev.* B **18** 3568–78
Svetitsky B 1980 *PhD dissertation* (Princeton University)